Lecture 35

Planar Graphs (contd.)

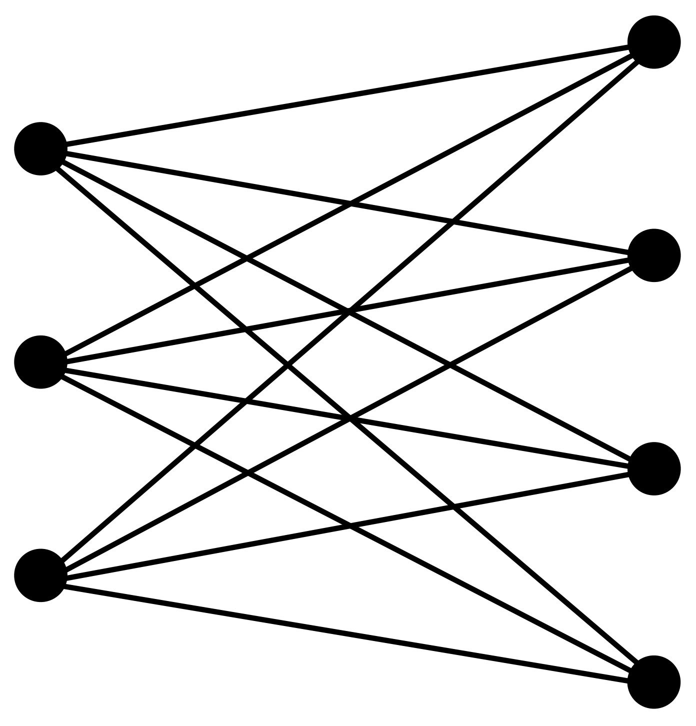
Complete Bipartite Graph

Definition: A **complete bipartite graph** is a graph whose vertices can be partitioned into two subsets V_1 and V_2 such that no edge has both the vertices (endpoints) in the same set and every possible edge that could connect vertices in different subsets is part of the graph.

A complete bipartite graph with partitions of size $|V_1| = m$ and $|V_2| = n$, is denoted $K_{m,n}$.

Example:

$K_{3,4}$



$K_{3,3}$ is Non-Planar

Theorem: $K_{3,3}$ is non-planar.

Proof: Suppose $K_{3,3}$ is planar. Since $K_{3,3}$ has 6 vertices and 9 edges, it must have 5 faces.

Let b_1, b_2, b_3, b_4, and b_5 be the boundaries of the 5 faces and m_1, m_2, m_3, m_4 and m_5 be the length of the boundaries, respectively, with respect to some planar embedding.

Then,

$$m_1 + m_2 + m_3 + m_4 + m_5 = 18 \quad (\because \textit{every edge appears twice on boundaries})$$

All $b_i s$ are cycles because there is no vertex of degree 1.

But cycles of length 3 are not possible because $K_{3,3}$ is a bipartite graph.

Hence, $m_i \geq 4$, which leads to a contradiction as $m_1 + m_2 + m_3 + m_4 + m_5 \geq 20$ ■

A Necessary Condition for Non-Planarity

Theorem: If G is a planar graph of $n \geq 3$ vertices and m edges, then $m \leq 3n - 6$.

Proof: Assume G is connected.

Let $f_1, f_2, f_3, \ldots, f_k$ be the faces of G (w.r.t an embedding) and m_i be the length of the boundary of the face f_i.

$$m_1 + m_2 + \ldots + m_k = 2m$$

$$3k \leq 2m \qquad (\because \textit{ each boundary's length } \geq 3)$$

$$3(2 - n + m) \leq 2m \qquad (\textit{from Euler's theorem})$$

$$m \leq 3n - 6$$

Will it not disturb planarity?

If the graph is not connected, connect it by adding more, say k, edges. Then,

$$m + k \leq 3n - 6 \implies m \leq 3n - 6$$

■

More on Planar Graphs

Corollary: Every planar graph contains a vertex of degree 5 or less.

Proof: Let G be a planar graph and suppose every vertex is of degree at least 6 or more.

Let n be the number of vertices and m be the number of edges.

Let d_i be the degree if the ith vertex. Then,

$$d_1 + d_2 + \ldots + d_n = 2m \implies 6n \leq 2m \implies 3n \leq m$$

But $m \leq 3n - 6$ for planar graphs. Hence, a contradiction. ■

Corollary: K_5 is non-planar.

Proof: It does not satisfy the $m \leq 3n - 6$ criteria for planarity. ■

Kuratowski's Theorem

Definition: A **subdivision** of a graph is a graph formed by subdividing its edges into paths of one or more edges.

Kuratowski's Theorem:

A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.